ON COMPACTIFICATION OF METRIC SPACES*

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ABSTRACT

If $f: X \to X^*$ is a homeomorphism of a metric separable space X into a compact metric space X^* such that $\overline{f(X)} = X^*$, then the pair (f, X^*) is called a metric compactification of X. An absolute G_δ-space (F_σ-space) X is said to be of the first kind, if there exists a metric compactification (f, X^*) of X such that $f(X) = \bigcap_{i=1}^{\infty} G_i$, where G_i are sets open in X^* and $\dim[Fr(G_i)] < \dim X$. ($Fr(G_i)$ being the boundary of G_i and $\dim X$ — the dimension of X). An absolute G_δ-space (F_σ-space), which is not of the first kind, is said to be of the second kind. In the present paper spaces which are both absolute G_δ and F_σ-spaces of the second kind are constructed for any positive finite dimension, a problem related to one of A. Lelek in [11] is solved, and a sufficient condition on X is given under which $\dim[X^* - f(X)] \geqq k$, for any metric compactification (f, X^*) of X, where $k \leqq \dim X$ is a given number.

Introduction. Let $f: X \to X^*$ be a homeomorphism of a *separable metric* space X into a compact *metric* space X^*, such that $\overline{f(X)} = X^*$. The pair (f, X^*) is then called a *metric compactification* of X. If X is an absolute G_δ-space (F_σ-space) (i.e. a G_δ-set (F_σ-set) in some compact space), then X is said to be of the first kind (cf. [6]) provided there exists a compactification (f, X^*) of X such that $f(X) = \bigcap_{i=1}^{\infty} G_i$, where G_i are sets open in X^* and $\dim [Fr(G_i)] < \dim X$, $i = 1, 2, \cdots$. ($Fr(G_i)$ denotes the boundary of G_i, and $\dim X$ the dimension of X in the sense of Menger-Urysohn.) An absolute G_δ-space (F_σ-space) which is not of the first kind is said to be of the second kind. The aim of the present paper is: (i) to construct, for any positive finite dimension, spaces X which are both absolute F_σ and absolute G_δ-spaces of the second kind; (ii) to solve a problem related to one of A. Lelek in [11]; and (iii) to give a sufficient condition on X, such that, for a given $k \leqq \dim X$, we have $\dim [X^* - f(X)] \geqq k$ for every compactification (f, X^*) of X.

The paper consists of four parts. In Section I some known compactifications are mentioned; in Section II several problems concerning compactifications are

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posed. Facts on coverings are quoted in Section III. Finally, Section IV contains a solution of the problems(1) posed in Section II.

I. Some Compactifications of Metric Spaces

I.1. Let X be a given topological space. Let $X^* = X \cup (x^*)$, where $x^* \notin X$ is an additional point, and let us define the toppology in X^* by taking as open sets all sets open in X and all subsets U of X^*, such that $X^* - U$ is a closed compact subset of X. Then the theorem of Alexandroff states:

(1) The space X^* is a compact topological space and X^* is a Hausdorff space if and only if X is a locally compact Hausdorff space(2).

The space X^* is called the one-point compactification of the space X.

A topological embedding is usually allowed rather than insist that X actually be a subset of X^*.

Thus by a compactification of a space X a pair (f, X^*) is understood, such that $f: X \to X^*$ is a homeomorphism of X into a compact space X^* and $\overline{f(X)} = X^*$ (i.e. the image $f(X)$ of X is dense in X^*). In this sense the one-point compactification of a non compact space X is a pair (i, X^*) where $i: X \to X^*$ is the identity mapping and $\overline{i(X)} = X^* = X \cup (x^*)$.

Another compactification of a topological space X is the Stone-Čech compactification $(e, \beta(X))$(3). This compactification is defined as follows:

Let us take the set $F(X)$ of all continuous functions $f: X \to J$ mapping X into the interval $J = [0, 1]$ and the product $J^{F(X)}$ with the Tychonoff topology. Let us define the mapping $e: X \to J^{F(X)}$ by correlating with each point $x \in X$ the point $e(x)$ whose f-th coordinate is $f(x)$, for each $f \in F(X)$. The mapping $e(x)$ is a continuous mapping of X into $J^{F(X)}$, and in the case when X is a completely regular T_1-space it turns out to be a homeomorphism. In this case we define $\beta(X)$ by by $\beta(X) = \overline{e(X)}$ and the pair $(e, \beta(X))$ is called the Stone-Čech compactification of X.

Let us note that:

(2) If $(e, \beta(X))$ is the Stone-Čech compactification of a completely regular T_1-space X and $f: X \to Y$ is a continuous mapping of X into a compact Hausdorff space Y, then $f[e^{-1}(x)]$ has a continuous extension on $\beta(X)$ into Y(4).

Numerous other compactifications were constructed for various purposes. One of them, used in the dimension theory, is the Wallman compactification $(\Phi, w(X))$. It turns out to be topologically equivalent to the Stone-Čech compactification provided $w(X)$ is a Hausdorff space(5).

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(1) I learned recently that some problems considered in the present study have been solved by A. Lelek in an entirely different way (not published).

(2) See [5], p. 150, also [3], p. 73.

(3) See [5], p. 152. For properties of the Stone-Čech compactification, see also [2] and [13].

(4) See [5], p. 153.

(5) Ibidem, p. 168. For properties of the Wallman compactification, [15].

I.2. Considering the one-point compactification (i, X^*) of a metric space, we note that the space X^* is generally not a metric space. For instance, if X is a metric space which is not locally compact, then by (1) X^* cannot be a metric space (since every metric space is a Hausdorff space). Thus if we seek, for a given metric space X, a compactification (f, X^*) where X^* is also a metric space, we generally cannot achieve this by merely adding a single point, and should allow the set $X^* - f(X)$ to contain more than one point.

In the present study we confine ourselves to metric compactifications (f, X^*) of metric separable spaces X only, i.e., we assume that *X is a separable, metric space and X^* a metric space.* As already noted, the one-point compactification is generally not a metric compactification. Let us show that an analogous statement holds for the Stone-Čech compactification $(e, \beta(X))$.

THEOREM 1. *If X is a non compact metric space and $(e, \beta(X))$ the Stone-Čech compactification of X, then $\beta(X)$ is not a metric space*[6].

Proof. Suppose, to the contrary, that $\beta(X)$ is a metric space. Let $e(X)$ be the image of X in $\beta(X)$. Since X is not compact, there exists a sequence $A = \{a_n\}_{n=1,2\,\dots}$ of points $a_n \in X$ which does not contain any convergent subsequence. Consider the points $e(a_n) = b_n$. Since $\beta(X)$ is compact and metric, the sequence $\{b_n\}_{n=1,2,\dots}$ contains a convergent subsequence $\{b'_n\} \subset \{b_n\}$. Let $b'_n \to b \in \beta(X)$ and consider the points $a'_n = e^{-1}(b'_n)$. By $A' = \{a'_n\} \subset A$ the sequence A' does not contain any convergent subsequence. Therefore A' is a closed subset of X. Let us define the real function $f: A' \to J = [0,1]$ by

$$f(a'_n) = \begin{cases} 0 & \text{for } n = 2k \\ 1 & \text{for } n = 2k-1 \end{cases} \quad k = 1, 2, \cdots .$$

Since A' does not contain any convergent subsequence, the function $f: A' \to J$ is continuous and since A' is a closed subset of the metric space X, we can, using Tietze's extension theorem[7], extend this function, to a continuous function $f: X \to J$ (the extended function is denoted also by f). By (2), the function fe^{-1} has a continuous extension $\tilde{f}$ to the whole of $\beta(X)$. But since

$$\tilde{f}(b'_n) = fe^{-1}(b'_n) = f(a'_n) = \begin{cases} 0 & \text{for } n = 2k \\ 1 & \text{for } n = 2k-1 \end{cases}$$

and $b'_n \to b$, the function $\tilde{f}$ cannot be continuous at the point b. This contradiction shows that $\beta(X)$ is not a metric space.

(6) This theorem seems to be well known. It was noted by A. Zabrodsky that the above proof may be applied to show that $\beta(X)$ can not even satisfy the first countability axiom.

(7) See [8], p. 117.

REMARK 1. Since the Wallman compactification $(\Phi, w(X))$ is topologically equivalent to that of Stone-Čech, provided $w(X)$ is a Hausdorff space it follows by Theorem 1 that if X is a non-compact metric space, then the space $w(X)$ is not a metric space.

II. PROBLEMS ON COMPACTIFICATIONS

II.1. The results of Section I indicate that metric compactifications of metric spaces are generally neither the Stone-Čech nor the one-point compactification. Now, since for metric compactifications the set $X^* - f(X)$ generally contains more than one point, there arises a problem of finding the structure of this set for some classes of metric spaces X. For example the following questions can be posed:

(a) Is it always possible to find a compactification (f, X^*) of X such that $X^* - f(X)$ would be countable?

(b) Is it always possible to find a compactification (f, X^*) such that $\dim [X^* - f(X)] < \dim X$?

Regarding question (a) it is known that each space which does not contain a subset dense in itself, has a compactification (f, X^*) such that $X^* - f(X)$ is countable(8). On the other hand, it is easily seen that for each compactification of the set X of rational numbers the set $X^* - f(X)$ is uncountable.

Indeed, since $f: X \rightarrow X^*$ is a homeomorphism, each point of $f(X)$ is a limit point and therefore X^* is perfect. Hence X^* is uncountable(9).

Regarding (b) it is known(10) that for each space X, there exists a compactification (f, X^*) such that $\dim X^* = \dim X$ and thus $\dim [X^* - f(X)] \leqq \dim X$. Easy examples show that in many cases this weak inequality $\leqq$ can be replaced by the strong $<$. It suffices, for example to take any n-dimensional cube J^n; $n = 1, 2, \cdots$ and any point $p \in J^n$. The set $X = J^n - (p)$ can be compactified by adding this single point. We then have $X^* = J^n$ and

$$\dim [X^* - f(X)] = \dim (p) = 0 < \dim X,$$

where $f = i$ is the identity mapping. On the other hand, it is not always possible to achieve the strong inequality $\dim(X^* - f(X)) < \dim X$. Indeed, for a 0-dimensional space X, $\dim (X^* - f(X)) < \dim X = 0$ means that $X^* - f(X)$ is empty and hence X is compact. It follows that for a 0-dimensional non compact space X this strong inequality is impossible. The problem of finding examples of n-dimensional spaces X, $n > 0$ of a simple topological structure for which $\dim [X^* - f(X)] < \dim X$ does not hold for any compactification (f, X^*) of X is more complicated. More precisely, this problem may be formulated as follows:

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(8) See [7], p. 194, IV.

(9) See [3], p. 98.

(10) See [4], p. 65, Theorem V, 6. Also [9], p. 72.

(c) Let X be a given n-dimensional space and $k \leqq n$ an integer. Under what conditions on X shall we have $\dim[X^* - f(X)] \geqq k$ for each compactification (f, X^*) of X?

II.2. B. Knaster discovered in [6] that there exist two kinds of absolute G_δ-spaces (also called G_δ-spaces in compact spaces or topologically complete spaces). Their definition is(11):

An absolute G_δ-space is said to be of the first kind, if there exists a compactification (f, X^*) such that $f(X) = \bigcap_{i=1}^{\infty} G_i$ and $\dim[Fr(G_i)] < \dim X$, where G_i, $i = 1, 2, \cdots$, are sets open in X^* and $Fr(G_i)$ denotes the boundary of G_i in X^*. An absolute G_δ-space is said to be of the second kind if it is not of the first kind.

It was shown by Lelek(12) that

(3) An absolute G_δ-space of finite dimension is of the first kind, if and only if there exists a compactification (f, X^*) of X such that $\dim[X^* - f(X)] < \dim X$.

Now, it was shown in [6] that the Cartesian product $N \times J$, where N is the set of irrational numbers in the interval $J = [0, 1]$, is an absolute G_δ-space of the second kind. It was further proved in [11] that if Z is any compact space with $\dim Z = n \geqq 0$, then the space $X = N \times Z$ is an absolute G_δ-space of the second kind. These results provide a solution of problem (c) for $n = k$ in the class of finite dimensional absolute G_δ-spaces. The sequel will include a solution of the following problems:

(a_1) Does there exist, for any positive finite dimension $n = 1, 2, \cdots$, a finite dimendsional space X, which is both an absolute F_σ and G_δ-space of the second kind?

(a_2) Is it true that each absolute G_δ-space X of the second kind, of positive finite dimension n, contains a topological image of a set of the form $N \times Z$, where N is the set of irrational numbers of the interval $J = [0, 1]$ and $\dim Z = \dim X$?

(a_3) Problem (c).

(a_4) Construction of a weakly infinite dimensional absolute F_σ and G_δ-space of the first kind such that for each compactification (f, X^*) there is $\dim(X^* - f(X)) = \infty$(13).

Before proceeding with a solution of problems (a_1)–(a_4), we quote in the next section some facts on coverings.

(11) See: Introduction

(12) See [11], p. 31, Theorem 1.

(13) A space is called weakly infinite-dimensional if it is a union of a sequence of finite dimensional spaces X_k, with $\dim X_k \to \infty$, for $k \to \infty$.

III. Coverings

A *covering* of a space Y is a family $\mathscr{G} = \{G_i\}$ of sets G_i such that $Y = \bigcup_i G_i$. If G_i are open (closed) sets, the covering is called open (closed). If the diameters $\delta(G_i)$ of all G_i are $< \varepsilon$, $\mathscr{G}$ is called an ε-covering, and if $\mathscr{G}$ is finite—a finite covering.

$d_n(Y)$ denotes the infinum of all numbers $\varepsilon > 0$ such that there exists a finite open ε-covering of Y satisfying

(4) $G_{i_0} \cap G_{i_1} \cap \cdots \cap G_{i_n} = \emptyset$, for any set of $n+1$ indices $i_0 < i_1 < \cdots < i_n$ (i.e., such that the intersection of any $n+1$ different sets G_i is empty).

It is known that for finite coverings of a space Y the existence of an open ε-covering satisfying (4) is equivalent to the existence of a closed ε-covering satisfying (4), and that for a compact space Y, $\dim Y \leqq n$ if and only if $d_{n+1}(Y) = 0$(14). Let us now prove a property of the Lebesgue number λ of a finite covering.

(5) Let $F_0, F_1, \cdots, F_m$ be a finite family of closed subsets of a compact space Z. Then there exists a number $\lambda > 0$ (the Lebesgue number of the family $(F_0, F_1, \cdots F_m)$) such that if there exists a point $p \in Z$ at distance $\leqq \lambda$ from all the sets $F_{k_0}, F_{k_1}, \cdots, F_{k_j}$, then $\cap_{i=0}^{j} F_{k_i} \neq \emptyset$.

Proof.(15). Suppose the contrary. Then there exists a sequence of points $p_n \in Z$, $n = 0,1,2,\cdots$, and families $S_j = \{F^j_{k_0}, \cdots, F^j_{k_{n_j}}\}$, $j = 0,1,2,\cdots$, of sets such that the point p_j is at distance $\leqq (1/(j+1))$ from all the sets $F_{k_i^j}$ of the family S_j, but $\cap_{i=0}^{n_j} F_{k_i^j} = \emptyset$. Since the number of different families S_j, $j = 0,1,\cdots$ constructed from a given finite family of sets $\{F_k\}_{k=0,1,\ldots m}$ is finite, some family —say S_0—must appear in the sequence $\{S_0\}_{j=0,1,\ldots}$ an infinite number of times. Thus there exists a subsequence $\{p_n'\} \subset \{p_n\}$ such that p_n' is at distance $\leqq (1/(n+1))$ from all the sets $F_{k_0^0}, \cdots, F_{k_{n_0}^0}$ of S_0. Since Z is compact, the sequence $\{p_n'\}$ contains a convergent subsequence to some point $p \in Z$. Denoting this subsequence by $\{p_n'\}$, we have $p_n' \to p \in Z$. Now, since $\rho(p_n', F_{k_i^0}) \leqq (1/(n+1))$ for $i = 0,1,\cdots,n_0$, and $n = 0,1,\cdots$, and since $p_n' \to p$, we have $\rho(p, F^0_{k_i}) = 0$. Thus $p \in F^0_{k_i}$, $i = 0,1,\cdots,n_0$, which is incompatible with the fact $\bigcap_{i=0}^{n_0} F_{k_i^0} = \emptyset$ (by the definition of S_j).

It follows by (5) that

(6) Let Y be a closed subset of a compact space Z and let $Y \subset \bigcup_{k=0}^{m} F_k$, where F_k are closed sets such that any different $n+1$ of them have an empty intersection. Replacing each F_k by its ε-neighborhood/(16) $G_k = S(F_k, \varepsilon)$ (in Z), where $2\varepsilon < \lambda$, we obtain an open (in Z) covering $\mathscr{G} = \{\bar{G}_k\}$ of Y, such that for the family $\{\bar{G}_k\}$ of closures of G_k, any $n+1$ different sets $\bar{G}_k$ have an empty intersection(17).

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(14) See [9], p. 60.

(15) This is a standard proof and is given here for the sake of completeness only.

(16) An ε-neighborhood of a set F is by definition the union over all $p \in F$ of the sets $S_p = [z; \varrho(p,z) < \varepsilon : z \in Z]$

(17) For a proof of (6) see also [14], p. 414, Lemma 2 and [10], p. 257.

Another consequence of (5) is;

(7) If the closed sets $F_0, F_1, \cdots, F_m$ in a compact space Z have an empty intersection then there exists a number $\varepsilon > 0$ such that no set of diameter $\leqq \varepsilon$ has a non empty intersection with each of the sets $F_0, F_1, \cdots, F_m$.

Indeed, it suffices to take $\varepsilon = \lambda$ and to apply (5).

We shall now give some properties of coverings of simplexes.

Let $\sigma_s = (p_0, \cdots, p_s)$ be a closed s-dimensional simplex with vertices $p_0, p_1, \cdots, p_s$ in the Euclidean s-dimensional space E^s and let $f: \sigma^s \rightarrow Z$ be a homeomorphism of σ^s into a space Z. Let $\sigma^{s-1,i}$ denote the $(s-1)$ dimensional closed face of σ^s opposite to the vertex $p_i \in \sigma^s$, i.e. $\sigma^{s-1,i} = (p_0, \cdots, p_{i-1}, p_{i+1}, \cdots, p_s)$, $i = 0, 1, \cdots, s$, and let $\tau^s = f(\sigma^s)$ and $\tau^{s-1,i} = f(\sigma^{s-1,i})$. Then τ^s is a curvilinear simplex with vertices $q_i = f(p_i)$ and $(s-1)$-dimensional faces $\tau^{s-1,i}$, $i = 0, 1, \cdots, s$. Since f is a homeomorphism and $\bigcap_{i=0}^{s} \sigma^{s-1\ i} = \emptyset$, we have that $\bigcap_{i=0}^{s} \tau^{s-1,i} \neq \emptyset$. Thus applying (7) with $m = s$ to the closed sets $F_i = \tau^{s-1,i}$, we find that there exists a number $\varepsilon > 0$ such that no set with diameter $\leqq \varepsilon$ intersects each of the faces $\tau^{s-1,i}$.

(8) Let $\varepsilon > 0$ be a number such that no set with diameter $\leqq \varepsilon$ intersects each face $\tau^{s-1,i}$. Let further $\tau^s = \bigcup_{k=0}^{m} F_k$, where F_k are closed sets with diameters $\delta(F_k) \leqq \varepsilon$, $k = 0, 1, \cdots, m$. Then some $s+1$ sets $F_{k_0}, \cdots, F_{k_s}$ have a non empty intersection.

Since $\delta(F_k) \leqq \varepsilon$, no F_k containing a vertex q_j of τ^s intersects the face $\tau^{s-1,j}$ opposite to q_j. Since f is one-to-one, no set $f^{-1}(F_k)$ containing a vertex p_j of σ^s intersects the face $\sigma^{s-1,j}$ opposite to p_j. Now, the sets $f^{-1}(F_k)$, $k = 0, 1, \cdots m$, cover the simplex σ^s and are closed, since f is continuous. Thus applying the same procedure as in the proof of [2, 24] in ([1], p. 194) we obtain that some $s+1$ sets $f^{-1}(F_{k_j})$, $j = 0, 1, \cdots s$, have a non empty intersection. Hence also the sets F_{k_j}, $j = 0, 1, \cdots s$, have a non empty intersection.

IV. SOLUTION OF THE PROBLEMS FORMULATED IN II

IV.1. *An n-dimensional absolute F_σ and G_δ-space X and its properties.*

Let $\sigma^n = (p_0, p_1, \cdots, p_n)$ be the n-dimensional closed simplex in the n-dimensional Euclidean space E^n with vertices $p_0 = (0, 0, \cdots, 0)$ and $p_i = (0, \cdots 0, 1, 0, \cdots 0)$, $i = 1, 2, \cdots, n$. (i.e. p_i is the point in E^n whose i-th coordinate is 1 and all i other coordinates are 0). Let $A = \{a_j\}$, $j = 1, 2, \cdots$, be the sequence of points of the form $a_j = (1/j)$, $j = 1, 2, \cdots$ on the real axis E^1 and let $a_0 = 0 \in E^1$. Denote by $Fr(\sigma^n) = \bigcup_{i=0}^{n} \sigma^{n-1,i}$ the boundary of the simplex σ^n. Let

$$(9) \qquad X = (A \times \sigma^n) \cup [(a_0) \times Fr(\sigma^n)]$$

Then $X \subset E^{n+1}$ and the closure $\bar{X}$ of X in E^{n+1} is

$$\bar{X} = (A \times \sigma^n) \cup [(a_0) \times \sigma^n] = [A \cup (a_0)] \times \sigma^n.$$

Since $\bar{X}$ is a compact subset of E^{n+1} (as a product of two compact spaces $A \cup (a_0)$ and σ^n), $\bar{X}$ is a compact space, and since X can be written as a union

$[(a_0) \times Fr(\sigma^n)] \cup [\bigcup_{j=1}^{\infty} (a_j) \times \sigma^n]$ of a countable number of compact sets, it follows that X is an absolute F_σ space.

On the other hand the set $\bar{X} - X$ equals the interior of the simplex $(a_0) \times \sigma^n$. Since this interior is a union of compact sets, the set $\bar{X} - X$ is an F_σ set and therefore X is a G_δ-set in $\bar{X}$. It follows that

(b_1) The set X defined in (9) is both an absolute F_σ and G_δ-space. Evidently, $\dim X = n$.

We shall now show that

(b_1') For each compactification (f, X^*) of X we have $\dim[X^* - f(X)] \geqq \dim X = n$.

Indeed, suppose to the contrary that $\dim [X^* - f(X)] \leqq n - 1 < \dim X$ and consider the sets $\tau_j^n = f[(a_j) \times \sigma^n]$, $j = 1, 2, \cdots$, and $\tau_j^{n-1,i} = f[(a_j) \times \sigma^{n-1,i}]$, $i = 0, 1, \cdots, n$, $j = 0, 1, \cdots$. Since $a_j \to a_0$ for $j \to \infty$, it follows that for every $i = 0, 1, \cdots, n$, $\operatorname{dist}\{[(a_j) \times \sigma^{n-1,i}], [(a_0) \times \sigma^{n-1,i}]\} \to 0$ for $j \to \infty$, where $\operatorname{dist}(A, B) = \max[\sup_{x \in A} \rho(x, B), \sup_{x \in B} \rho(A, x)]$ is the distance of the sets A and B in the sense of Hausdorff(18). Since $f : X \to X^*$ is continuous and $[A \cup (a_0)] \times \sigma^{n-1,i}$ is compact it follows easily that

$$\text{(10)} \qquad \operatorname{dist}(\tau_j^{n-1,i}, \tau_0^{n-1\,i}) \to 0 \text{ for } j \to \infty \text{ and each } i = 0, 1, \cdots, n.$$

Now, the space X^* being compact, there exists a subsequence $\{j'\}$ of $\{j\}$ such that the sequence of sets $\{\tau_{j'}^n\}$ converges to a continuum $C \subset X^*$(19). Writing j instead of j', we have $\operatorname{dist}(\tau_j^n, C) \to 0$ for $j \to \infty$. If there were $C \cap [\bigcup_{j=1}^{\infty} \tau_j^n] \neq \emptyset$, then there would exist a point y_0 and a sequence $y_{j_k} \in \tau_{j_k}^n$ of points, such that $y_{j_k} \to y_0 \in \tau_{j_0}^n$ for $k \to \infty$ and some j_0. Then $x_{j_k} = f^{-1}(y_{j_k}) \to f^{-1}(y_0) = x_0$, which is, because of $x_{j_k} \in (a_{j_k}) \times \sigma^n$ and $x_0 \in (a_{j_0}) \times \sigma^n$, incompatible with the openness of $(a_{j_0}) \times \sigma^n$ in the union $\bigcup_{j=1}^{\infty} (a_j) \times \sigma^n$. It follows that $C \cap [\bigcup_{j=1}^{\infty} \tau_j^n] = \emptyset$, and since the set $\cup_{i=0}^{n} \tau_0^{n-1,i}$ is an $(n-1)$-dimensional compact subset of C, it follows from the assumption $\dim[X^* - f(X)] \leqq n - 1$ and from Corollary 1 in ([4], p. 32), that $\dim C \leqq n - 1$. Thus, by the definition of $d_n(Y)$ (cf. section III), we obtain $d_n(C) = 0$. Hence, by (6), there exists for every $\varepsilon > 0$ an ε-covering of C by sets G_k open in X^*, $k = 0, 1, \cdots, m$ such that

$$\text{(11)} \quad \bar{G}_{k_0} \cap \bar{G}_{k_1} \cap \cdots \cap \bar{G}_{k_n} = \emptyset \text{ for any set of subscripts } k_0 < k_1 < \cdots < k_n.$$

Since $\bigcap_{i=0}^{n} \tau_0^{n-1,i} = \emptyset$ we may, according to (7), choose for this covering an ε so small that no $\bar{G}_k$ intersects each set $\tau_0^{n-1,i}$. Hence by (10) no set $\bar{G}_k$ intersects all the faces $\tau_j^{n-1,i}$, $i = 0, 1, \cdots, n$, for sufficiently large j. Let $G = \bigcup_{k=0}^{m} G_k$. Since $C \subset G$ and $\operatorname{dist}(\tau_j^n, C) \to 0$ for $j \to \infty$, there exists a j_0 such that $\tau_j^n \subset G$ for $j \geqq j_0$. Fixing any $j \geqq j_0$, we find that the sets $F_k = \tau_j^n \cap \bar{G}_k$, $k = 0, 1, \cdots, m$, satisfy the assumptions of (8) with s replaced by n and τ by τ_j. Hence by (8) some $n + 1$ sets $F_{k_0}, \cdots, F_{k_n}$, and therefore also the sets $\bar{G}_{k_0}, \cdots, \bar{G}_k$ have a non empty intersection, which is incompatible with (11). Thus (b_1') is proved.

(18) See [8], p. 106

(19) See [9], p. 110. Also [16], p. 11.

From (b_1), (b_1') and (3) we obtain

THEOREM 2. *The set X defined in* (9) *is both an absolute F_σ and G_δ-space of the second kind and of dimension n.*

This theorem gives an answer to problem (a_1).

IV. 2. **On a problem of A. Lelek.** The following problem P. 313 was formulated by Lelek in [11], p. 34).

Does there exist, for each absolute G_δ-space X of the second kind with finite, positive dimension, a compact space Z with positive dimension, such that X contains a topological image of the set $N \times Z$ (N being the set of irrational numbers of the interval $J = [0,1]$)?

A negative answer to this question was given in [12]. Now it is easily seen that a negative answer to problem (a_2) posed in section II contains, as a special case, a negative answer to the problem of Lelek. (It suffices to take, in (a_2), $n = \dim X = 1$.) We now proceed to prove that the answer to (a_2) is negative.

Indeed, let X be the space defined in (9). We shall show that there does not exist a space Z with $\dim Z = \dim X = n$ such that $N \times Z$ has a topological image in X.

Suppose, to the contrary, that such a space Z exists and let $h: N \times Z \to X$ be a homeomorphism of $N \times Z$ into X. Fix a point $\xi \in N$. Then the n-dimensional space $(\xi) \times Z$ has a topological image in X. Now X being a countable union of compact disjoint sets $(a_j) \times \sigma^n$ and $(a_0) \times Fr(\overset{n}{\sigma})$, $j = 1,2,\cdots$ and $(\xi) \times Z$ being n-dimensional,it follows[20] that $h[(\xi) \times Z]$ has an n-dimensional intersection with some set $(a_{j(\xi)}) \times \sigma^n$. This intersection, as n-dimensional subset of σ^n, contains[21] an open subset of $(a_{j(\xi)}) \times \sigma^n$. Since h is one-to-one, the sets $h[(\xi) \times Z]$ and $h[(\xi') \times Z]$ are disjoint for $\xi \neq \xi', \xi, \xi' \in N$, and since N is uncountable, we get an uncountable family of disjoint open sets contained in X, which is impossible.

IV. 3. **Two theorems on compactification.** We shall now prove two theorems which will enable us to provide an answer to problem (c) and to construct, for any $n = 1,2,\cdots,\aleph_0$, a n-dimensional space X which is not locally compact at a *single* point and such that for each compactification (f, X^*) of X we heva $\dim[X^* - f(X)] \geqq 1$.

THEOREM 3. *Suppose that the space X contains a sequence $\{C_i\}_{i=1,2,\dots}$ of continua C_i and a point p such that*

(c_1) the sets C_i are closed and open in the union $\bigcup_{i=1}^{\infty} C_i$ and $C_i \cap C_j = \emptyset$ for $i \neq j$;

(20) This is a consequence of the Sum Theorem for Dimension n, Cf. [4], p. 30.

(21) This follows easily from Theorem IV, 3 in [4], p. 44.

(c_2) there exists a number $\delta > 0$, such that $\delta(C_i) \geqq \delta$ for each $i = 1, 2, \cdots$;

(c_3) $\overline{\bigcup_{i=1}^{\infty} C_i} - \bigcup_{i=1}^{\infty} C_i = (p)$.

Then X is not locally compact at the point p, and for each compactification (f, X^*) of X we have $\dim[X^* - f(X)] \geqq 1$.

Proof. Let U_p be an arbitrary neighborhood containing the point p. We have to show that the closure $\bar{U}_p$ is not compact. By (c_1) and (c_3) there exists a sequence of points $p_i \in \bigcup_{i=1}^{\infty} C_i$ such that $p_i \to p$ for $i \to \infty$ and such that the sequence $\{p_i\}_{i=1,2,\dots}$ has only a finite number of points in common with each C_i. Thus we may assume that for each $i = 1, 2, \cdots$, we have $p_i \in C_i$. Let $S = S(p, r)$ be a spherical neighborhood of p with radius $r < \delta/2$ contained in U_p. Since $p_i \to p$, the sets $C_i \cap S$ are not empty for i sufficiently large, and since C_i are connected, we obtain from (c_2) that for these i; $C_i \cap Fr(S) \neq \emptyset$. Choose from each such set $C_i \cap Fr(S)$ a point q_i and consider the sequence $\{q_i\}$. Since $\bar{S} \subset \bar{U}_p$, we have $\{q_i\} \subset \bar{U}_p$ and since $q_i \in Fr(S)$, it follows that $\rho(q_i, p) = r > 0$. Now, since $q_i \in C_i$ for i sufficiently large, (c_1) and (c_3) imply that any convergent subsequence of $\{q_i\}$ tends to p, which is impossible because $\rho(q_i, p) = r > 0$. Thus $\bar{U}_p$ is not compact. It remains to show that if (f, X^*) is any compactification of X, then $\dim[X^* - f(X)] \geqq 1$. For this purpose let us consider the sets $X_1 = \bigcup_{i=1}^{\infty} C_i \cup (p)$ and $f(X_1)$. The closure $\overline{f(X_1)} = X_1^* \subset X^*$ is a compactification of X_1. Let y be any point of $X_1^* - f(X_1)$. Then the point $y \notin f(X)$. Indeed, if there would exist a point $x \in X$ such that $y = f(x)$, then we would have $x \notin X_1$, since f is one-to-one. Now, $y \in \overline{f(X_1)}$ implies that there exists a sequence of points $x_n \in X_1$ such that $f(x_n) \to y$. By the continuity of f^{-1} we have $x_n \to x \in X - X_1$. But by (c_3) the set X_1 is closed in X, and since $x_n \in X_1$ it follows that $x \in X_1$. This contradiction shows that $y \notin f(X)$. Thus

$$(12) \qquad [X_1^* - f(X_1)] \cap f(X) = [\overline{f(X_1)} - f(X_1)] \cap f(X) = \emptyset.$$

Let us take further $r < \delta/2$ and construct (in analogy with the first part of the proof) points $p_i \to p$, $p_i \in C_i$ and $q_i \in C_i$, such that $\rho(p, q_i) = r > 0$ for i sufficiently large. Since $X_1^* = \overline{f(X_1)}$ is compact and $f(C_i) \subset X_1^*$ we can choose a subsequence of the sequence $\{f(C_i)\}$ of continua converging([22]) to some continuum C. Denoting the subscpripts of this subsequence by i we have therefore that $\operatorname{dist}[f(C_i), C] \to 0$ for $i \to \infty$. Now, since $p_i \to p$ and $p_i \in C_i$, it follows that C contains the point $f(p)$. If C would reduce to this point $f(p)$, then $q_i \in C_i$ would imply $f(q_i) \to f(p)$, and since f^{-1} is continuous we would also have $q_i \to p$, in contradiction to $\rho(p, q_i) = r > 0$. It follows that C contains at least two points, and since it is a continuum we have $\dim C \geqq 1$. Therefore $\dim[C - (f(p)] \geqq 1$.

([22]) See [9], p. 110.

Now, by (c_1) we have $C \cap f(C_i) = \emptyset$ for each $i = 1,2,\cdots$. Therefore $X_1^* \subset X^*$ and (12) imply that $\dim[X^* - f(X)] \geqq 1$. Theorem 3 is proved.

EXAMPLE 1. Let $X = (a_0) \cup [\bigcup_{j=1}^{\infty} (a_j) \times J]$ where $a_0 = 0$ and $a_j = 2^{-j+1}$ $j = 1,2,\cdots$, are real numbers on the real axis and $J = [0,1]$ (Figure 1). This 1-dimensional space X is not locally compact at the single point $a_0 = 0$, and by Theorem 3 $\dim[X^* - f(X)] \geqq 1$ for any compactification (f, X^*) of X. It is also easily seen that X is an absolute F_σ and G_δ-space and thus, by (3) and $\dim X = 1$, we obtain that X is an absolute F_σ and G_δ-space of the second kind.

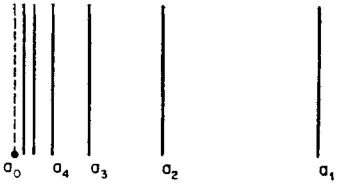

Figure 1

EXAMPLE 2. Let $n = 2,3,\cdots,\aleph_0$, and let $X = (J^n - X_1) \cup (O)$, where $X_1 = \{x;\ x = (x_1, x_2, \cdots, x_n),\quad x_1 = 0,\quad 0 \leqq x_i \leqq 1,\quad \text{for}\quad i = 2,3,\cdots,n\}$ and $O = (0,0,\cdots,0)$. (If $n = \aleph_0, J^n$ is the Hilbert cube). It is clear that $\dim X = n$, and that X is not locally compact at the single point O. It is also easy to construct a sequence C_i of continua in X, such that the assumptions of Theorem 3 be satisfied for the pooint $p = O$. Hence $\dim[X^* - f(X)]] \geqq 1$ for any compactification (f, X^*) of X (for $n = 3$, see Figure 2).

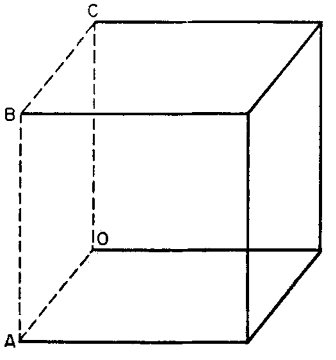

Figure 2

By Theorem 3 for each compactification (f,X^*) of this full cube X excluding the full square $OABC$ but including point O, $\dim\ [X^* - f(X)] \geqq 1$.

Let us now show that

(d) If the set X is a closed subset of space Y and for each compactification (g, X^*) of X we have $\dim[X^* - g(X)] \geqq k$, then for each compactification (f, Y^*) of Y we have $\dim[Y^* - f(Y)] \geqq k$.

Proof. The closure (in Y^*) $\overline{f(X)} = X^*$ of $f(X)$ is a compactification (f, X^*) of X and therefore by assumption, we have $\dim[X^* - f(X)] \geqq k$. Now, it is easily seen that $\overline{f(X)} \cap f(Y - X) = \emptyset$. Indeed, otherwise we could find a point $x_0 \in Y - X$ and a sequence of points $x_n \in X$ such that $f(x_n) \to f(x_0)$. But since f is homeomorphism on Y there would be $x_n \to x_0$, which is incompatible with the closedness of X in Y. From $\overline{f(X)} \cap f(Y - X) = \emptyset$, we obtain

$$X^* - f(X) \subset Y^* - f(Y),$$

and therefore $\dim[Y^* - f(Y)] \geqq k$.

As a consequence of (b$'_1$) and (d), we have the following answer to problem (c):

THEOREM 4. If space Y contains topologically the set X defined in (9) and X is a closed subset of Y, then for each compactification (f, Y^*) of Y^* we have $\dim[Y^* - f(Y)] \geqq n$.

(The case $n = 2$ is illustrated in Figure 3).

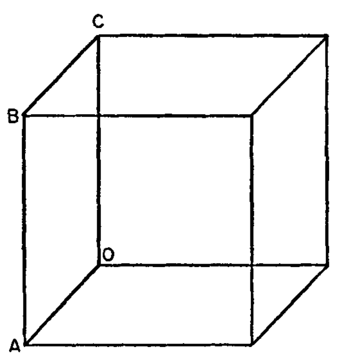

Figure 3

According to theorem 4, for each compactification (f, Y^*) of this full cube Y excluding the interior of the square $OABC$ (but including OA, AB, BC and CO) $\dim[Y^* - f(Y)] \geqq 2$.

IV. 4. A weakly infinite-dimensional absolute F_σ and G_δ-space.

As stated in (3), a finite dimensional absolute G_δ-space X is of the first kind if and only if there exists a compactification (f, X^*) of X such that $\dim[X^* - f(X)] < \dim X$.

We shall now show that the above condition is not necessary for infinite dimensional spaces. More precisely, we shall construct an absolute F_σ and G_δ-space of the first kind which is weakly infinite-dimensional and such that for each compactification (f, X^*) of X we have $\dim[X^* - f(X)] = \infty$. Let us take, for fixed n, the set A_n of points $x_{n\,m} = 2^{-n} + 2^{-m}$, $m = n+1, n+2, \cdots$, on the real axis. Define $X_n = (A_n \times \sigma^n) \cup [(2^{-n}) \times Fr(\sigma^n)]$. where σ^n is an n-dimensional closed simplex with diameter $\delta(\sigma^n) = 2^{-n}$. Let $X = \bigcup_{n=1}^{\infty} X_n$.

The set X can be considered as a subset of the Hilbert cube $J^{\aleph_0}$, and its closure $\bar{X}$ is $\bar{X} = \bigcup_{n=1}^{\infty} X_n \cup [\bigcup_{n=1}^{\infty} (2^{-n}) \times \mathrm{Int}(\sigma^n)] \cup (O)$ where $\mathrm{Int}\,\sigma^n = \sigma^n - Fr(\sigma^n)$ and $O = (0, 0, \cdots)$ is the point all whose coordinates are zero. It is also easily seen that $\bar{X}$ may be written in the form $\bigcup_{n=1}^{\infty} \bar{X}_n \cup (O)$, where $\bar{X}_n = [A_n \cup (2^{-n})] \times \sigma^n$. Since $\bar{X}$ is a compact space and X is a countable union of compact sets, we find that X is an absolute F_σ-space. Further, we can write each set $(2^{-n}) \times \mathrm{Int}(\sigma^n)$ as a union $\bigcup_{i=1}^{\infty} F_i^n$ of compact sets F_i^n, $i = 1, 2, \cdots$. Thus

$$\bar{X} - X = \bigcup_{n=1}^{\infty} \bigcup_{i=1}^{\infty} F_i^n \cup (O)$$

is an F_σ set and thus X is an absolute G_δ-space. Moreover, the sets

$$\bar{X} - \left[\bigcup_{n=1}^{s} \bigcup_{i=1}^{s} F_i^n \cup (O) \right] = G_s$$

are open in $\bar{X}$, $\dim[Fr(G_s)] \leqq s$ and $\bigcap_{s=1}^{\infty} G_s = X$. Hence, X is an absolute F_σ and G_δ-space of the first kind. By the definition of X, it follows that X is a weakly infinite-dimensional space i.e. $\dim X = \infty$(22).

We shall now show that for each compactification (f, X^*) of X we have $\dim[X^* - f(X)] = \infty$. For this purpose, let us note that the set X_n is homeomorphic with the space defined in (9), and hence by (b$'_1$) we have

$$\dim[X_n^* - f(X_n)] \geqq \dim X_n = n$$

for each compactification (f, X_n^*) of X_n. Now it is easily seen that X_n is a closed subset of X. Thus, applying (d) for $X = X_n$ and $Y = X$, we have

$$\dim[X^* - f(X)] \geqq n.$$

Since n is arbitrary, it follows that $\dim[X^* - f(X)] = \infty$.

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(22) For weakly infinite-dimensional spaces X, $\dim X = \omega$ is sometimes written instead of $\dim X = \infty$.

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